1

Introduction

- In this topic, we will
  - Describe what will be covered in the next topic
    - Approximating solutions to boundary-value problems (BVPs)
  - Review how a BVP differs from an initial-value problem
  - Describe Dirichlet and Neumann boundary conditions
  - Give a description of linear ordinary differential equations
  - Introduce the following two topics

2

Solutions to equations

- In your calculus course, you have been introduced to:
  - Initial-value problems (IVPs)
  - Boundary-value problems (BVPs)
- A 2nd-order BVPs involves a 2nd-order ordinary differential equation (ODE) and two conditions at two different points
  - Generally, we are looking for a solution as a function of space, so we will use a function $u(x)$ and not $y(t)$
  - The constraints or *conditions* will be specified at two points:

$$x = a \text{ and } x = b$$

- The conditions for a BVP can include:
  - Fixed or Dirichlet conditions
  - Neumann conditions

3

3

Dirichlet boundary conditions

- The most common boundary conditions are *fixed* or Dirichlet boundary conditions:

$$u(a) = u_a$$
$$u(b) = u_b$$

  - These specify the value of the function at two different points

4

4

Neumann boundary conditions

- Alternatively, we may specify the derivative at one of the boundary points

$$u^{(1)}(a) = u_a^{(1)}$$

  - Such a condition is described as a *Neumann boundary condition*
  - If the derivative is zero, this indicates an insulated boundary:

$$u^{(1)}(a) = 0$$

  - A non-zero derivative may model a boundary point that dissipates power at a known rate

What is an insulated boundary?

- Why does a zero derivative imply an insulated boundary?
  - If your hand is 10 cm from a block of ice,
    you expect it to get colder if you move closer to it
  - If your hand is 10 cm from a source of heat,
    you expect it to get warmer if you move closer to it
  - If your hand is 10 cm from an insulated wall,
    you don't expect the temperature to change greatly if you move it closer to that wall

What is an insulated boundary?

- An example: mukluks

User: Daderot

7

7

Approximating solutions to boundary-value problems

Looking ahead

- We shall see two techniques:
  - The shooting method
  - Finite-difference method
- There is an additional third technique:
  - The finite element method
- We'll discuss the first two, and introduce the third

8

8

Definitions

- A 2nd-order linear ordinary differential equation (LODE) is one that is linear in the unknown function and its derivatives
  - The coefficients and right-hand side can be functions of *x*

$$a_2(x)u^{(2)}(x)+a_1(x)u^{(1)}(x)+a_0(x)u(x)=g(x)$$

  - If the forcing function is 0, we call it a *homogenous* LODE:

$$a_2(x)u^{(2)}(x)+a_1(x)u^{(1)}(x)+a_0(x)u(x)=0$$

  - If the coefficients are constant, LODE *with constant coefficients*:

$$a_2u^{(2)}(x)+a_1u^{(1)}(x)+a_0u(x)=g(x)$$

    - The forcing function need not be constant
  - Of course, we can have a *homogenous* LODE *with constant coefficients*:

$$a_2u^{(2)}(x)+a_1u^{(1)}(x)+a_0u(x)=0$$

Approximating solutions to boundary-value problems

Linear and non-linear equations

- In your calculus course, the only problems you are able to easily solve are those with constant coefficients

$$a_2u^{(2)}(x)+a_1u^{(1)}(x)+a_0u(x)=0$$

- However, if an ODE is not a LODE, it can still, in general, be described in the form

$$u^{(2)}(x)=f\left(x,u(x),u^{(1)}(x)\right)$$

Looking ahead

- The shooting method will work on an arbitrary 2nd-order ODE
- The finite-difference method will only work for LODEs
  - The techniques we see for the second will lead us to techniques we will use in approximating solutions to partial-differential equations
    - Specifically, Laplace's equation, the heat-conduction equation and the wave equation

11

11

Approximating solutions to boundary-value problems

Summary

- Following this topic, you now
  - Have an overview of the ideas to be covered in this sub-section
  - Understand the difference between Dirichlet and Neumann boundary conditions as well as the concept of an insulated boundary condition
  - Are aware of the two techniques we will consider
  - Understand the definitions of linear ODEs and LODEs with constant coefficients as well as the concept of a homogenous LODE

12

12

References

[1] https://en.wikipedia.org/wiki/Boundary_value_problem

Approximating solutions to boundary-value problems

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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15

15

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16

16